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Translated by J. J. D.

UDC 532.593 + 532.5:627

WAVES GENERATED BY PERTURBATIONS OF THE BOTTOM OF A TANK WITH A DOCK

PMM Vol. 36, №4, 1972, pp. 636-640
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(Received July 14, 1971)

A wave motion generated on the surface of a heavy incompressible fluid by oscillations of a section of the bottom of a tank with a dock is studied. The problem of waves generated by an oscillating section of the bottom of a tank was dealt with in [1, 2]. In the present paper the Wiener-Hopf method [3] is employed to solve the analogous problem in which the boundary conditions have been altered, namely, a part of the free surface is covered with an immovable rigid plate. An expression for the velocity potential describing the motion of the fluid in the problem under consideration is derived. The results of [2, 4, 5] are found to be particular cases of the solution obtained here. The numerical example given shows that the rise of the free surface is smaller on the dock side than that at the corresponding point at the side opposite to the oscillating section of the bottom.

1. An immovable rigid plate is situated at the surface of a fluid of finite depth h, occupying the region $y = h,\ x \leqslant -l$ and $-\infty < z < \infty$. The coordinate origin is placed at the bottom of the tank and the y-axis is directed vertically upwards. The section $y = 0,\ 0 \leqslant x \leqslant a,\ -\infty < z < \infty$ of the bottom undergoes vertical displacement according to the law

$$y = \operatorname{Re}[v(x) \exp i(kz - \omega t)]$$

where $v(x)$ is a numerically small, smooth function. The velocity potential $F(x, y, z, t)$ which in this case describes the motion of the fluid, must satisfy the following boundary value problem

$$\Delta F(x, y, z, t) = 0 \quad (0 \leqslant y \leqslant h,\ -\infty < x < \infty,\ -\infty < z < \infty)$$

$$\partial^2 F / \partial t^2 + g \partial F / \partial y = 0 \ \text{when}\ y = h, \quad x > -l, \quad -\infty < z < \infty$$

$$\partial F / \partial y = 0 \ \text{when}\ y = h, \quad x \leqslant -l, \quad -\infty < z < \infty \tag{1.1}$$

$$\partial F / \partial y = \begin{cases} -\omega \operatorname{Re}[iv(x) \exp i(kz - \omega t)] & (0 \leqslant x \leqslant a) \\ 0 \quad (-\infty < x < 0,\ a < x < \infty) \end{cases} y = 0, \quad -\infty < z < \infty$$

The fluid motion must be bounded near the point $(-l, h)$ as well as at a distance from the dock, and it must gradually decay under the dock. The first condition is expressed by the requirement that $\partial F / \partial t$ is bounded at the dock edge [1].

The function $F(x, y, z, t)$ is sought in the form

$$F(x, y, z, t) = \operatorname{Re}[\varphi(x, y) \exp i(kz - \omega t)] \qquad (1.2)$$

For $\varphi(x, y)$ we have

$$\Delta\varphi - k^2\varphi = 0 \qquad (0 \leqslant y \leqslant h, \ -\infty < x < \infty)$$

$$\partial\varphi / \partial y - \beta\varphi = 0 \text{ when } y = h, \quad x > -l \quad (\beta = \omega^2 / g)$$

$$\partial\varphi / \partial y = 0 \text{ when } y = h, \quad x \leqslant -l$$

$$\partial\varphi / \partial y = \begin{cases} -i\omega v(x) & (0 \leqslant x \leqslant a) \\ 0 & (-\infty < x < 0, \ a < x < \infty) \end{cases} \text{ when } y = 0 \qquad (1.3)$$

$$|\varphi(x, y)| < M = \text{const} \text{ when } r = [(x + l)^2 + (y - h)^2] \to 0$$

$$\lim_{x \to \infty} \varphi = D_+ \operatorname{ch} C_0 y e^{i\vartheta x}, \quad D_+ = \text{const}, \quad \vartheta = \text{const}$$

$$\lim_{x \to -\infty} \varphi = D_- \exp kx, \quad k > 0, \quad D_- = \text{const}$$

where $\pm iC_0$ are roots of the equation $\beta \cos Ch + C \sin Ch = 0$. Applying to (1.3) the Fourier transforms and making use of the notation

$$\Phi(\alpha, y) = \int_{-\infty}^{\infty} \varphi(x, y) e^{i\alpha x} dx, \qquad \alpha = \sigma + i\tau$$

$$\Phi_+(\alpha, y) = \int_{-l}^{\infty} \varphi(x, y) e^{i\alpha(x+l)} dx, \qquad V(\alpha) = \int_0^a v(x) e^{i\alpha x} dx$$

$$\Phi_-(\alpha, y) = \int_{-\infty}^{-l} \varphi(x, y) e^{i\alpha(x+l)} dx, \qquad \gamma^2 = \alpha^2 + k^2$$

we obtain the following functional equation:

$$\Phi_+(\alpha, h) + K(\alpha)\Phi_-(\alpha, h) = \frac{i\omega V(\alpha) e^{i\alpha l}}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h} \qquad (1.4)$$

$$K(\alpha) = \frac{\gamma \operatorname{sh} \gamma h}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h} \qquad (0 < \tau < k, \ -\infty < \sigma < \infty)$$

Here $\Phi_+(\alpha, h)$ is a function regular in the semiplane $\tau > 0$ and $\Phi_-(\alpha, h)$ is regular in the semiplane $\tau < k$. The kernel $K(\alpha)$ of (1.4) is a function, regular in the strip under consideration.

$$\Phi(\alpha, y) = \frac{i\omega V(\alpha)}{\gamma} \frac{\gamma \operatorname{ch} \gamma(h - y) - \beta \operatorname{sh} \gamma(h - y)}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h} - \frac{\beta e^{-i\alpha l} \Phi_-(\alpha, h) \operatorname{ch} \gamma y}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h} \qquad (1.5)$$

2. The functional equation (1.4) is solved using the Wiener-Hopf method [3]. Let us factorize the function [4]

$$K_+(\alpha) = \frac{i\alpha - k}{\beta[1 - h^2(\alpha^2 + k^2) / \rho_0^2]} \prod_{n=1}^{\infty} \frac{[1 + k^2h^2 / (n^2\pi^2)]^{1/2} - i\alpha h / (n\pi)}{[1 + k^2h^2 / \rho_n^2]^{1/2} - i\alpha h / \rho_n}$$

$$K_-(\alpha) = \frac{1}{h(k+i\alpha)} \prod_{n=1}^{\infty} \frac{[1+k^2h^2/\rho_n^2]^{1/2}+i\alpha h/\rho_n}{[1+k^2h^2/(n^2\pi^2)]^{1/2}+i\alpha h/(n\pi)}$$

where $\pm \rho_0 / h$ and $\pm i\rho_n / h$ are roots pf the equation $\rho \operatorname{sh} \rho h - \beta \operatorname{ch} \rho h = 0$, and $\rho_n = n\pi + \beta h / (n\pi)$ when $n \gg 1$. Multiplying (1.4) by $[K_+(\alpha)]^{-1}$ we obtain

$$\frac{\Phi_+(\alpha, h)}{K_+(\alpha)} + \frac{\Phi_-(\alpha, h)}{K_-(\alpha)} = \frac{i\omega\, V(\alpha)\, e^{i\alpha l}}{[\gamma \operatorname{sh}\gamma h - \beta \operatorname{ch}\gamma h]\, K_+(\alpha)} = i\omega\Gamma(\alpha)$$

Applying now the partition method [3], the Liouville theorem [4] and taking into account the condition (1.3) at the edge, we obtain the following relation for the above equation:

$$\frac{\Phi_+(\alpha, h)}{K_+(\alpha)} - \frac{\omega}{2\pi} \int_{ic-\infty}^{ic+\infty} \frac{\Gamma(\xi)\, d\xi}{\xi - \alpha} = -\frac{\Phi_-(\alpha, h)}{K_-(a)} - \frac{\omega}{2\pi} \int_{id-\infty}^{id+\infty} \frac{\Gamma(\xi)\, d\xi}{\xi - \alpha} = P = \text{const}$$

$$(0 < c < \tau < d < k)$$

from which follows

$$\Phi_-(\alpha, h) = -K_-(\alpha)\left[P + \frac{\omega}{2\pi} \int_{id-\infty}^{id+\infty} \frac{\Gamma(\xi)\, d\xi}{\xi - \alpha}\right] \tag{2.1}$$

The integral in (2.1) should be computed using the residues at the poles of the upper semiplane, since the integrand expression contains, as a multiplier, the function $[K_+(\xi)]^{-1}$ which is analytic in the upper semiplane and has an infinite number of poles in the lower semiplane. The integrand expression in (2.1) has, in addition to the roots of $[V(\xi)]^{-1}$, the following singularities (poles) in the upper semiplane:

$$\xi = i\nu_n, \quad \nu_n = [k^2 + \rho_n^2 / h^2]^{1/2} \quad (n = 1, 2, \ldots)$$

Using the theorem of residues, we can write (2.1) in the form

$$\Phi_-(\alpha, h) = -K_-(\alpha)\Big[\sum \operatorname{res}(\xi_*) + P + \tag{2.2}$$

$$+\, \omega \sum_{n=1}^{\infty} \frac{\rho_n^2 V(i\nu_n)\, e^{-\nu_n l}}{\nu_n h\, [\beta^2 h^2 - \beta h + \rho_n^2] \cos \rho_n\, (i\nu_n - \alpha)\, K_+(i\nu_n)}\Big]$$

Here ξ_* are the roots of the equation $[V(\xi)]^{-1} = 0$. Now, applying the inverse Fourier transforms to (1.5) and taking (2.2) into account, we obtain the solution of (1.3).

3. Let the amplitude function be given by $v(x) = \varepsilon \sin \pi/a\, x$, where ε denotes the maximum deflection of the points lying on the median of the oscillating section of the bottom and is assumed small compared with h. Then

$$V(\alpha) = \varepsilon \int_0^a \sin \frac{\pi}{a} x e^{i\alpha x}\, dx = \varepsilon \frac{\pi}{a} \frac{1 + e^{i\alpha a}}{\pi^2 / a^2 - \alpha^2} \tag{3.1}$$

The denominator in the right-hand part of (3.1) does not vanish in the upper semiplane $\tau > 0$, therefore in the present case (2.2) assumes the form

$$\Phi_-(\alpha, h) = -K_-(\alpha) \times$$

$$\times \left[P + \varepsilon\omega \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{(1 + e^{-\nu_n a})\, \rho_n^2 e^{-\nu_n l}}{\nu_n h\, [\beta^2 h^2 - \beta h + \rho_n^2] \cos \rho_n\, (\nu_n^2 + \pi^2 / a^2)\, (i\nu_n - \alpha)\, K_+(i\nu_n)}\right] \tag{3.2}$$

Applying the inverse Fourier transform to (1.5) and taking into account (3.1) and (3.2), we can write the solution of (1.3) for our particular case, as

$$\varphi(x, y) = \Psi(x, y) + i\varepsilon\omega\frac{\pi}{a}\Big\{f_0(y)[\sin\delta(x-a) + \sin\delta x] + \sum_{n=1}^{\infty} f_n(y)(1 + e^{\nu_n a})e^{-\nu_n x}\Big\} \text{ when } a < x < \infty \tag{3.3}$$

$$\varphi(x, y) = i\varepsilon\omega\frac{\pi}{a}\Big\{\frac{a[b\,\mathrm{ch}\,b(h-y) - \beta\,\mathrm{sh}\,b(h-y)]}{\pi b(b\,\mathrm{sh}\,bh - \beta\,\mathrm{ch}\,bh)}\sin\frac{\pi}{a}x + f_0(y)\sin\delta x + \sum_{n=1}^{\infty} f_n(y)[e^{-\nu_n x} - e^{\nu_n(x-a)}]\Big\} + \Psi(x, y) \text{ when } 0 \leqslant x \leqslant a \tag{3.4}$$

$$\varphi(x, y) = \Psi(x, y) + i\varepsilon\omega\frac{\pi}{a}R(x, y) \text{ when } -l < x < 0 \tag{3.5}$$

$$\varphi(x, y) = i\varepsilon\omega\frac{\pi}{a}\Big\{\beta\sum_{n=1}^{\infty}\frac{A_n(1 + e^{-\nu_n a})}{(\nu_n^2 + \pi^2/a^2)\sin\rho_n}\cos\frac{\rho_n}{h}y e^{\nu_n x} + R(x, y)\Big\} + \beta\Big\{\sum_{n=1}^{\infty}(-1)^n\frac{K_+(i\mu_n)}{\mu_n}\psi_-(i\mu_n)e^{\mu_n(x+l)}\cos\frac{n\pi}{h}y + \psi_-(ik)\frac{K_+(ik)}{2kh}e^{k(x+l)}\Big\} \text{ when } -\infty < x < -l \tag{3.6}$$

$$f_0(y) = \frac{2A_0}{\delta^2 - \pi^2/a^2}\Big[\frac{\rho_0}{h}\mathrm{ch}\frac{\rho_0}{h}(h-y) - \beta\,\mathrm{sh}\frac{\rho_0}{h}(h-y)\Big]$$

$$f_n(y) = \frac{A_n}{\nu_n^2 + \pi^2/a^2}\Big[\frac{\rho_n}{h}\cos\frac{\rho_n}{h}(h-y) - \beta\sin\frac{\rho_n}{h}(h-y)\Big]$$

$$A_0 = \frac{\rho_0}{\delta(\beta h - \beta^2 h^2 + \rho_0^2)\,\mathrm{ch}\,\rho_0}, \qquad A_n = \frac{\rho_n}{\nu_n(\beta^2 h^2 - \beta h + \rho_n^2)\cos\rho_n}$$

$$b = \Big(k^2 + \frac{\pi^2}{a^2}\Big)^{1/2}, \qquad \Psi(x, y) = -i\beta\Big\{A_0\frac{\rho_0}{h}[\psi_-(\delta)K_-(\delta)e^{-i\delta(x+l)} - \psi_-(-\delta)K_-(-\delta)e^{i\delta(x+l)}]\,\mathrm{ch}\frac{\rho_0}{h}y + i\sum_{n=1}^{\infty}A_n\frac{\rho_n}{h}\psi_-(-i\nu_n)K_-(-i\nu_n)e^{-\nu_n(x+l)}\cos\frac{\rho_n}{h}y\Big\}$$

$$R(x, y) = -\sum_{n=1}^{\infty} f_n(y)(1 + e^{-\nu_n a})e^{-\nu_n x}$$

$$\psi_-(\alpha) = -\frac{\Phi_-(\alpha, h)}{K_-(\alpha)}$$

It should be noted that the formulas (3.3) – (3.6) yield the results of [2] for $l \to \infty$, the results of [4] for $\varepsilon \to 0$, and the results of [5] for $l \to 0$.

4. If we limit ourselves to the case $l/h \gg 1$, i.e. if we assume that the distance between the plate edge and the oscillating section of the bottom exceeds the depth of the fluid, then the terms containing the factor $\exp(-\nu_n l)$ will vanish from the expressions (3.3)–(3.6) and the form of the surface will be given by the formulas

$$\zeta(x, z, t) = \varepsilon\pi\beta\{B_0[\sin\delta x + \sin\delta(x-a)] + \sum_{n=1}^{\infty} B_n(1+e^{\nu_n a})e^{-\nu_n x}\}\cos(kz-\omega t)$$
$$\text{when } a < x < \infty \qquad (4.1)$$

$$\zeta(x, z, t) = \varepsilon\pi\beta\left\{\frac{a\sin\pi x/a}{\pi(b\,\mathrm{sh}\,bh - \beta\,\mathrm{ch}\,bh)} + B_0\sin\delta x + \sum_{n=1}^{\infty} B_n[e^{-\nu_n x} - e^{\nu_n(x-a)}]\right\}\times$$
$$\times\cos(kz-\omega t) \quad \text{when } 0 \leqslant x \leqslant a \qquad (4.2)$$

$$\zeta(x, z, t) = -\varepsilon\pi\beta\sum_{n=1}^{\infty} B_n(1+e^{-\nu_n a})e^{\nu_n a}\cos(kz-\omega t) \qquad (4.3)$$
$$\text{when } -l < x < 0$$

$$B_0 = \frac{2A_0\rho_0/h}{\delta^2 - \pi^2/a^2}, \qquad B_n = \frac{A_n\rho_n/h}{\nu_n^2 + \pi^2/a^2}$$

Formulas (4.1)–(4.3) do not include the terms which are obtained from (3.3)– (3.6) in the absence of motion of the bottom ($\varepsilon = 0$), i.e. the terms which define the natural oscillations of the free surface of the fluid which are the same for all three regions $(-l, 0)$, $[0, a]$ and (a, ∞).

For the values of parameters $\varepsilon = 0.1$ m, $h = 1$ m, $a = 2$ m, $\omega = 4.34$ sec and $k =$ 1 m, we obtain the following values for the rise of the free surface ζ (m) along x (m): $x = -2$, 1 and 4 we have $\zeta = 0.001$, 0.194 and 0.174.

From these results we can conclude that the presence of a dock at the surface of a fluid exerts a stabilizing influence on small perturbations appearing as the result of a section of the bottom undergoing a deformation.

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Translated by L. K.